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On the existence of non-radiating frequencies in the radiation from a stochastic current distribution

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Abstract. The theory of non-radiating stochastic current distributions is developed within the framework of first-order coherence theory of electromagnetic fields. Necessary and sufficient conditions for non-radiating current correlations are derived. A class of non-radiating stochastic current distributions is constructed. In contrast to these results, it can be shown that any non-trivial field correlation given on an arbitrary surface radiates.

1. Introduction

It is well known (Sommerfeld 1904, 1905, Herglotz 1908, Ehrenfest 1910, Schott 1933, Goedecke 1964, Erber and Prastein 1970, Devaney and Wolf 1973, Cohen and Bleistein 1977) that oscillating charge current distributions with finite support may not radiate, but rather lead to a static field outside the distribution.

Much of the older work on non-radiating current distributions was stimulated by problems connected with extended electron models, models for elementary particles, or electromagnetic self-force and radiation reaction (see e.g. Jackson (1974) and Hoenders (1978) for a review).

The current renewed interest in non-radiating sources is motivated by inverse scattering and source reconstruction problems, and in particular with the question of the uniqueness of the reconstruction (see Hoenders (1978) for a review): in general, current distributions cannot be determined uniquely from their radiation pattern since non-radiating distributions exist.

Such distributions can always be added to the source without changing the radiation pattern. In other words, a current distribution can only be determined up to its non-radiating part.

A famous example of a non-radiating distribution has been constructed by Schott (1933) who considered a surface charge distribution e, uniformly distributed at the surface of a sphere with radius a. In Schott's own words: When the centre of a uniformly and rigidly charged sphere, with charge e and radius a, in purely translatory motion, describes a closed orbit periodically in a time $2a/cj$, where j is any integer, the electromagnetic field at every point is a static field. The electrostatic potential of this field is the same as that due to a charge e distributed along the orbit of the centre with a linear density varying inversely as the velocity of the centre, which is the same as that of

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every point of the sphere, and the magnetic field is the same as that due to a uniform steady current of strength $je/2a$ flowing round the orbit of the centre in the direction of its motion.

A general class of non-radiating current charge distributions, which allow a simple physical interpretation, has been derived by Ehrenfest (1910). Let ϕ be a solution of the Laplace equation in a domain a which is the exterior of a bounded domain b, and suppose that ϕ is time-independent in a, but time-dependent in b, that $\nabla^2\phi \neq 0$ in b, and that both ϕ and its normal derivative are continuous across the boundary of a and b. Then, if

$$\boldsymbol{E} = -\nabla\phi \qquad \nabla^2\phi = \rho \qquad \boldsymbol{j} = \rho\boldsymbol{v} \tag{1.1}$$

$$\boldsymbol{v} = \frac{(\partial/\partial t)\nabla\phi}{\nabla^2\phi} \qquad \text{everywhere in } a \cap b$$

we observe that the field (1.1) is a solution to Maxwell's equations if

$$\partial H/\partial t = 0$$

and

$$\nabla \times \boldsymbol{H} = 0 \qquad \text{everywhere in } a. \tag{1.2}$$

Therefore, the field generated by the current charge distribution defined in (1.1) is static outside the domain a and is therefore non-radiating. (This field is also the *unique* solution to Maxwell's equations if we impose the vectorial form of Sommerfeld's radiation condition.)

Equation (1.1) shows that for this, most general type of non-radiating current charge distributions, conjectured by Ehrenfest, *the convection current* $\boldsymbol{j} = \rho\boldsymbol{v}$ *is cancelled by the displacement current.*

It is both important and interesting to notice from this procedure that *neither symmetries, nor the vanishing of multipole moments, etc*, are needed for the construction of a non-radiating distribution, and that the absence of a time-dependent field outside the distribution is due to a *complicated interference mechanism.*

Another example connected with non-radiating sources has been constructed by Kay and Moses (1956) (see also Hoenders (1978)). They showed that it is possible to construct a plane variable index of refraction $n^2(x)$, extending from $-\infty$ to $+\infty$, and approaching unity at $-\infty$ and $+\infty$ such that a plane wave at a fixed frequency and polarisation incident at all angles from $-\infty$ will be transmitted to $+\infty$ without reflection. To be more specific, let

$$\psi(x, y, z) = \exp(\mathrm{i}k_y y + \mathrm{i}k_z z)u(x) \tag{1.3}$$

be a solution of

$$(\nabla^2 + k^2 n^2(x))\psi(x, y, z) = 0. \tag{1.4}$$

Then, if

$$E \equiv (k_y^2 + k_z^2) \qquad \text{and} \qquad V(x) = k^2(1 - n^2(x)), \tag{1.5}$$

combination of equations (1.3), (1.4) and (1.5) leads to

$$(\mathrm{d}^2/\mathrm{d}x^2 + E - V(x))u(x) = 0 \qquad -\infty < x < +\infty. \tag{1.6}$$

Kay and Moses showed the existence of an infinite number of continuous and everywhere negative 'potentials' $V(x)$ such that

$$\begin{aligned} &u(x)\sim\exp(\mathrm{i}\sqrt{E}\,x) && \text{if } x\to-\infty \\ &u(x)\sim t(E)\exp(\mathrm{i}\sqrt{E}\,x) && \text{if } x\to+\infty \\ &|t(E)|=1. \end{aligned} \tag{1.7}$$

This result, namely equations (1.3), (1.4), and (1.7), can be interpreted in terms of non-radiating current distributions if we use a result obtained by Bromwich (1919). The field vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ of an arbitrary electromagnetic field in an *inhomogeneous isotropic* medium without free currents and charges can be derived by differentiation from two potentials, satisfying equation (1.4).

Therefore, the result (1.7), together with equations (1.3) and (1.4), shows that the induced currents and charges in the medium are not radiating.

Moreover, it is readily observed that equation (1.6) denotes the one-dimensional time-independent Schrödinger equation. We therefore come to the following quantum-mechanical interpretation of equations (1.3)–(1.7): it is possible to construct an infinity of potentials $V(x)$, extending from $-\infty$ to $+\infty$ such that every incoming plane wave with an arbitrary angle of incidence will be transmitted without reflection.

Though the pertinent theory has been well developed for deterministic current distributions and sufficient and necessary conditions have been established for deterministic sources to be non-radiating, no such results are known for the case of stochastic sources. In this paper we develop the theory of non-radiating stochastic currents. In § 3 we construct a class of non-radiating (first-order) current correlations. In § 4 we derive a criterion which allows us to check whether a given source correlation is non-radiating. In § 5 we show that non-trivial field correlations which are prescribed on an arbitrary surface always radiate. Since non-radiating coherent (deterministic) sources exist, one may presume that the related interference effect also applies to partially coherent (stochastic) sources of finite, but non-zero, coherence area such as black body radiation. These effects would be absent only in the hypothetical case of zero coherence area.

2. Summary of electrodynamic relations

In this paper we focus our attention on *spatial* fluctuations. We thus consider only one frequency component of the pertinent correlation functions. Technically this is equivalent to considering spatially stochastic, but time harmonic fields throughout this section and §§ 3–5 of this paper. In § 6 we touch upon the generalisation of our results to more general time-dependent sources.

The following classical relations are used in our calculations. From Maxwell's equation in MKS units

$$\begin{aligned} &\nabla\times\boldsymbol{H}=\boldsymbol{j}-\mathrm{i}k\boldsymbol{D} && \nabla.\boldsymbol{D}=\rho && \boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}(\boldsymbol{r})\exp(-\mathrm{i}\omega t) \\ &\nabla\times\boldsymbol{E}=\mathrm{i}k\boldsymbol{B} && \nabla.\boldsymbol{B}=0 && \boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r})\exp(-\mathrm{i}\omega t) \\ &\omega=ck \end{aligned} \tag{2.1}$$

and the material equations

$$\boldsymbol{D}=\epsilon\boldsymbol{E}\qquad \boldsymbol{B}=\mu\boldsymbol{H} \tag{2.2}$$

where ϵ and μ are constants, one has the vectorial wave equations

$$\begin{aligned}\nabla\times\nabla\times\boldsymbol{E}-k^2\epsilon\mu\boldsymbol{E}&=\mathrm{i}k\mu\boldsymbol{j}\\ \nabla\times\nabla\times\boldsymbol{H}-k^2\epsilon\mu\boldsymbol{H}&=\nabla\times\boldsymbol{j}.\end{aligned} \tag{2.3}$$

The related tensorial Green function $\mathscr{G}$ defined by

$$\mathscr{G}(\boldsymbol{r},r';k)=(\mathbb{1}+k^{-2}\nabla\nabla)G(\boldsymbol{r},r';k) \tag{2.4}$$

where

$$G(\boldsymbol{r},r';k)=\frac{\exp(\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{2.5}$$

$\mathbb{1}$ denotes the unit dyadic, and $\nabla\nabla$ is a dyadic differential operator, is a solution of the equation:

$$(\nabla\times\nabla\times-k^2)\mathscr{G}(\boldsymbol{r},\boldsymbol{r}'k)=\mathbb{1}\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{2.6}$$

The unique solutions to equations (2.3) satisfying the vectorial form of Sommerfeld's radiation condition are obtained from equation (2.6) and the tensor analogue of Green's theorem for well-behaved vectors $\boldsymbol{P}$ and dyadics $\mathscr{Q}$, namely:

$$\int_V\{\nabla\times\nabla\times\boldsymbol{P}.\mathscr{Q}-\boldsymbol{P}.\nabla\times\nabla\times\mathscr{Q}\}\,\mathrm{d}\tau=\int_\sigma\{(\boldsymbol{n}\times\nabla\times\boldsymbol{P}).\mathscr{Q}+(\boldsymbol{n}\times\boldsymbol{P}).(\nabla\times\mathscr{Q})\}\,\mathrm{d}\sigma \tag{2.7}$$

with V denoting a three-dimensional domain bounded by the surface σ. This relation is easily obtained by applying the corresponding vector formula (Morse and Feshbach 1953) to every column vector of the tensor $\mathscr{Q}$ (see appendix 2).

If the current $\boldsymbol{j}$ is confined to a volume τ, equations (2.3), (2.6) and (2.7) with $\mathscr{Q}=\mathscr{G}$ and $\boldsymbol{P}=\boldsymbol{E}$ or $\boldsymbol{B}$ and Sommerfeld's vectorial radiation condition, by virtue of which the surface integrals tend to zero if σ tends to infinity, lead to:

$$\boldsymbol{E}(\boldsymbol{r})=\mu\int_\tau\mathscr{G}(\boldsymbol{r},\boldsymbol{r}';k).\mathrm{i}k\boldsymbol{j}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' \tag{2.8}$$

$$\boldsymbol{B}(\boldsymbol{r})=\mu\int_\tau\mathscr{G}(\boldsymbol{r},\boldsymbol{r}';k).\nabla'\times\boldsymbol{j}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'. \tag{2.9}$$

Integration by parts of equations (2.8) and (2.9) leads to

$$\boldsymbol{E}(\boldsymbol{r})=\mathrm{i}k\mu\int_\tau G(\boldsymbol{r},\boldsymbol{r}';k)[\mathbb{1}+k^{-2}\nabla'\nabla'].\boldsymbol{j}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' \tag{2.10}$$

and

$$\boldsymbol{B}(\boldsymbol{r})=\mu\int_\tau G(\boldsymbol{r},\boldsymbol{r}';k)[\mathbb{1}+k^{-2}\nabla'\nabla'].\nabla'\times\boldsymbol{j}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'. \tag{2.11}$$

3. General construction of a non-radiating stochastic source

Instead of the above deterministic field variables we now consider the corresponding correlations. Let $\boldsymbol{L}(\boldsymbol{r})$ denote the six vector with components $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{r})$ and consider the first-order spatial correlation tensor

$$\mathscr{J}(\boldsymbol{r}_1,\boldsymbol{r}_2)\equiv\langle\boldsymbol{L}(\boldsymbol{r}_1)\boldsymbol{L}^*(\boldsymbol{r}_2)\rangle \tag{3.1}$$

for positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ outside the domain τ, where the brackets $\langle \ldots \rangle$ denote an ensemble average and $\boldsymbol{L}(\boldsymbol{r}_1)\boldsymbol{L}^*(\boldsymbol{r}_2)$ has to be read as the direct or dyadic product. The source of $\mathscr{I}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is a stochastic current distribution, confined to a volume τ, defined by the current correlation tensor

$$\mathscr{J}(\boldsymbol{r}_1, \boldsymbol{r}_2) \equiv \langle \boldsymbol{j}(\boldsymbol{r}_1)\boldsymbol{j}^*(\boldsymbol{r}_2)\rangle \equiv \mathrm{Tr}(\rho \boldsymbol{j}\boldsymbol{j}^*). \tag{3.2}$$

Physical examples are currents in a turbulent plasma or the currents induced by the incident field in a scattering experiment. We recall that the ensemble average can be defined using a statistical operator ρ and reading the electromagnetic quantities as field operators, namely $\langle \boldsymbol{L}(\boldsymbol{r}_1)\boldsymbol{L}^*(\boldsymbol{r}_2)\rangle \equiv \mathrm{Tr}\{\rho \boldsymbol{L}(\boldsymbol{r}_1)\boldsymbol{L}(\boldsymbol{r}_2)\}$, etc. Examples for ρ are the coherent or Glauber state, which would bring us back to the deterministic case, and thermal equilibrium. We have not to specify ρ in the following theory.

A stochastic current distribution is called non-radiating if

$$\mathscr{I}(\boldsymbol{r}_1, \boldsymbol{r}_2) = 0 \qquad \text{if} \qquad \boldsymbol{r}_1 \text{ and } \boldsymbol{r}_2 \notin \tau. \tag{3.3}$$

As will be shown in § 4, the definition (3.3) can be relaxed as follows. An equivalent condition for (3.3) is that the tensor $\mathscr{I}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ vanishes at positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in the far zone.

In the following we construct a simple example of a class of non-radiating stochastic current distributions. Suppose that the tensor $\mathscr{h}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is a four times continuously differentiable function of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, such that

$$\mathscr{h}(\boldsymbol{r}_1, \boldsymbol{r}_2) = 0 \qquad \text{if} \qquad \boldsymbol{r}_1 \text{ and } \boldsymbol{r}_2 \notin \tau. \tag{3.4}$$

Consider the current correlation tensor

$$\mathscr{J}_h(\boldsymbol{r}_1, \boldsymbol{r}_2) \equiv \langle \boldsymbol{j}(\boldsymbol{r}_1)\boldsymbol{j}^*(\boldsymbol{r}_2)\rangle_h \tag{3.5}$$

defined by

$$\mathscr{J}_h(\boldsymbol{r}_1, \boldsymbol{r}_2) = (\nabla_1^2 + k^2)(\nabla_2^2 + k^2)\mathscr{h}(\boldsymbol{r}_1, \boldsymbol{r}_2). \tag{3.6}$$

Combination of equation (2.10), (3.5) and (3.6) leads to

$$\begin{aligned}\langle \boldsymbol{E}(\boldsymbol{r}_1)\boldsymbol{E}^*(\boldsymbol{r}_2)\rangle \\ &= \int_\tau \int_\tau G(\boldsymbol{r}_1, \boldsymbol{r}_1'; k)G^*(\boldsymbol{r}_2, \boldsymbol{r}_2'; k)(\mathbb{1} + k^{-2}\boldsymbol{\nabla}_1'\boldsymbol{\nabla}_1')(\mathbb{1} + k^{-2}\boldsymbol{\nabla}_2'\boldsymbol{\nabla}_2') \\ &\quad \times (\nabla_1^2 + k^2)(\nabla_2^2 + k^2) . \mathscr{h}(\boldsymbol{r}_1', \boldsymbol{r}_2')\, \mathrm{d}\boldsymbol{r}_1', \mathrm{d}\boldsymbol{r}_2'.\end{aligned} \tag{3.7}$$

Using Green's theorem, from equation (3.7) we obtain

$$\begin{aligned}\langle \boldsymbol{E}(\boldsymbol{r}_1)\boldsymbol{E}^*(\boldsymbol{r}_2)\rangle \\ &= \mathscr{H}(\boldsymbol{r}_1, \boldsymbol{r}_2) + \int_\sigma \int_\sigma \left(G(\boldsymbol{r}_1, \boldsymbol{\sigma}_1; k)\frac{\partial}{\partial n_1} - \frac{\partial}{\partial n_1} G(\boldsymbol{r}_1, \boldsymbol{\sigma}_1; k)\right) \\ &\quad \times \left(G^*(\boldsymbol{r}_2, \boldsymbol{\sigma}_2; k)\frac{\partial}{\partial n_2} - \frac{\partial}{\partial n_2} G^*(\boldsymbol{r}_2, \boldsymbol{\sigma}_2; k)\right)\mathscr{H}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)\, \mathrm{d}\boldsymbol{\sigma}_1\, \mathrm{d}\boldsymbol{\sigma}_2\end{aligned} \tag{3.8}$$

where

$$\mathscr{H}(\boldsymbol{r}_1, \boldsymbol{r}_2) = (\mathbb{1} + k^{-2}\boldsymbol{\nabla}_1\boldsymbol{\nabla}_1)(\mathbb{1} + k^{-2}\boldsymbol{\nabla}_2\boldsymbol{\nabla}_2) . \mathscr{h}(\boldsymbol{r}_1, \boldsymbol{r}_2) \tag{3.9}$$

and where σ denotes a surface enclosing both the volume and the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. However, equations (3.4), (3.8) and (3.9) show that $\langle \boldsymbol{E}(\boldsymbol{r}_1)\boldsymbol{E}^*(\boldsymbol{r}_2)\rangle$ is zero if both $\boldsymbol{r}_1$ and

r_2 are positions outside τ. It can be shown in the same way that the correlation tensors $\langle \boldsymbol{E}(\boldsymbol{r}_1)\boldsymbol{H}^*(\boldsymbol{r}_2)\rangle$ and $\langle \boldsymbol{H}(\boldsymbol{r}_1)\boldsymbol{H}^*(\boldsymbol{r}_2)\rangle$ are zero if $\boldsymbol{r}_1$ and $\boldsymbol{r}_2 \notin \tau$. Thus we have derived the result that the correlation tensor $\mathscr{I}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ (3.1) generated by the current correlations (3.2) is identically zero at any position $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ outside the source domain τ. In particular we learn that the average energy density

$$\langle U(\boldsymbol{r})\rangle \sim \mathrm{Tr}\{\langle \boldsymbol{E}(\boldsymbol{r})\boldsymbol{E}^*(\boldsymbol{r})\rangle + \langle \boldsymbol{H}(\boldsymbol{r})\boldsymbol{H}^*(\boldsymbol{r})\rangle\} \tag{3.10}$$

as well as the average Poynting vector

$$\langle \boldsymbol{S}(\boldsymbol{r})\rangle \sim \langle \boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r})\rangle \tag{3.11}$$

vanish everywhere outside the source domain τ. Thus there is no energy flow through any closed surface which does not intersect the domain τ.

We have shown above that equations (3.4) and (3.6) constitute a recipe for obtaining non-radiating stochastic current correlations. Under appropriate differentiability conditions one can also show that a given non-radiating stochastic source correlation can always be written in the form (3.4) and (3.6). To this end we have to show that the special type of construction of a non-radiating distribution considered in § 3 does always apply to a given non-radiating distribution. This is easily achieved by choosing as $\mathscr{k}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ of the construction (3.6) the tensor

$$\mathscr{k}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle \boldsymbol{E}(\boldsymbol{r}_1)\boldsymbol{E}^*(\boldsymbol{r}_2)\rangle. \tag{3.12}$$

The representation (3.6) is therefore both sufficient and necessary for vanishing field correlations outside the support of the current distribution.

4. A necessary and sufficient condition for non-radiating sources

Let us now derive a more tangible necessary and sufficient condition for non-radiating current correlations. To this end, we replace the Green function in equations (2.10) and (2.11) by the far-zone approximation

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2; k) \simeq r^{-1} \exp(\mathrm{i}kr) \exp(-\mathrm{i}k\boldsymbol{s}\cdot\boldsymbol{r}) \tag{4.1}$$

where $\boldsymbol{s} = \boldsymbol{r}/r$ with $r = |\boldsymbol{r}|$ denotes the unit vector in the direction of the observation. The corresponding far-field correlation tensor becomes, on using (2.10), (2.11) and (4.1), for large values of $|\boldsymbol{r}_1|$ and $|\boldsymbol{r}_2|$:

$$\mathscr{I}(\boldsymbol{r}_1, \boldsymbol{r}_2) \simeq \frac{\exp[\mathrm{i}k(r_1 - r_2)]}{r_1 r_2} \int_\tau \int_\tau \mathrm{d}\boldsymbol{r}_1' \, \mathrm{d}\boldsymbol{r}_2' \, \exp[-\mathrm{i}k(\boldsymbol{r}_1'\cdot\boldsymbol{s}_1 - \boldsymbol{r}_2'\cdot\boldsymbol{s}_2)]\mathscr{K}(\boldsymbol{r}_1', \boldsymbol{r}_2'; \boldsymbol{s}_1, \boldsymbol{s}_2) \tag{4.2}$$

where

$$\mathscr{K}(\boldsymbol{r}_1', \boldsymbol{r}_2'; \boldsymbol{s}_1, \boldsymbol{s}_2) \equiv$$

$$\begin{pmatrix} \langle(\mathbb{1} - \boldsymbol{s}_1\boldsymbol{s}_1)\cdot\boldsymbol{j}(\boldsymbol{r}_1')(\mathbb{1} - \boldsymbol{s}_2\boldsymbol{s}_2)\cdot\boldsymbol{j}^*(\boldsymbol{r}_2')\rangle & \langle(\mathbb{1} - \boldsymbol{s}_1\boldsymbol{s}_1)\cdot\boldsymbol{j}(\boldsymbol{r}_1')(\mathbb{1} - \boldsymbol{s}_2\boldsymbol{s}_2)\cdot(k\boldsymbol{s}_2 \times \boldsymbol{j}^*(\boldsymbol{r}_2'))\rangle \\ \langle(\mathbb{1} - \boldsymbol{s}_1\boldsymbol{s}_1)\cdot(k\boldsymbol{s}_1 \times \boldsymbol{j}(\boldsymbol{r}_1'))(\mathbb{1} - \boldsymbol{s}_2\boldsymbol{s}_2)\cdot\boldsymbol{j}^*(\boldsymbol{r}_2)\rangle & \begin{matrix}\langle(\mathbb{1} - \boldsymbol{s}_1\boldsymbol{s}_1)\cdot(k\boldsymbol{s}_1 \times \boldsymbol{j}(\boldsymbol{r}_1'))(\mathbb{1} - \boldsymbol{s}_2\boldsymbol{s}_2) \\ \cdot(k\boldsymbol{s}_2 \times \boldsymbol{j}^*(\boldsymbol{r}_2'))\rangle\end{matrix} \end{pmatrix} \tag{4.3}$$

with $\boldsymbol{s}_1 = \boldsymbol{r}_1/r_1$, $\boldsymbol{s}_2 = \boldsymbol{r}_2/r_2$. In the 6×6 tensor (4.3) we have

$$(\mathbb{1} - \boldsymbol{s}_1\boldsymbol{s}_1)\cdot\boldsymbol{j}(\boldsymbol{r}_1') = -\boldsymbol{s}_1 \times \boldsymbol{s}_1 \times \boldsymbol{j}(\boldsymbol{r}_1')$$

etc. The term $\langle(\mathbb{1}-s_1s_1)\cdot j(r_1')(\mathbb{1}-s_2s_2)\cdot j^*(r_2')\rangle$ arises from the correlations $\langle E(r_1)E^*(r_2)\rangle$, whereas the term $\langle(\mathbb{1}-s_1s_1)\cdot j(r_1')(\mathbb{1}-s_2s_2)\cdot(ks_2\times j^*(r_2'))\rangle$ arises from the correlations $\langle E(r_1)B^*(r_2)\rangle$, etc.

Let now $\tilde{j}(ks)$ denote the Fourier transform of the stochastic current $j(r)$, and let us perform the integrations over r_1 and r_2 in equation (4.2). We then observe that the correlations generated by $\langle E(r_1)E^*(r_2)\rangle$ can only vanish if

$$\langle \tilde{j}^{\mathrm{T}}(ks_1)\tilde{j}^{\mathrm{T}*}(ks_2)\rangle = 0 \tag{4.4}$$

for all directions s_1 and s_2, where, by definition,

$$\tilde{j}^{\mathrm{T}}(ks) = s\times s\times \tilde{j}(ks). \tag{4.5}$$

Recalling that

$$(\mathbb{1}-s_ls_l)\cdot(s_l\times\tilde{j}(ks_l)) = -s_l\times s_l\times s_l\times\tilde{j}(ks_l) = s_l\times\tilde{j}(ks_l), \qquad l=1,2, \tag{4.6}$$

we observe that all the other correlations occuring in equation (4.2) like

$$\langle E(r_1)H^*(r_2)\rangle \simeq \langle \tilde{j}^{\mathrm{T}}(ks_1)s_2\times\tilde{j}^{\mathrm{T}}(ks_2)\rangle \tag{4.7}$$

can only vanish if (4.4) holds true.

We now show that the condition (4.4) is also *sufficient*, i.e. current correlations obeying equation (4.4) are non-radiating (see definition (3.3)). To this end we rewrite equation (4.4) using Bauer's expansion (Watson 1966)

$$\exp(-\mathrm{i}ks\cdot r) = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}(-\mathrm{i})^l Y_l^m(\alpha,\beta)Y_l^{m*}(\theta,\phi)j_l(kr) \tag{4.8}$$

where j_l denotes the spherical Bessel function of order l; α, β denote the polar angles of the vector s, and θ, ϕ those of r. Inserting the expansion (4.8) into condition (4.4) and using the linear independence of the spherical harmonics Y_l^m, we obtain an equivalent discrete set of conditions, namely:

$$c_{l,l',m,m'} \equiv \int_\tau\int_\tau \mathrm{d}r_1'\,\mathrm{d}r_2'\,Y_l^m(\theta_1',\phi_1')Y_{l'}^{m'*}(\theta_2',\phi_2')j_l(kr_1')j_l(kr_2')\mathcal{K}^{(1)}(r_1',r_2') = 0 \tag{4.9}$$

for all l, l', m, m', where

$$\mathcal{K}^{(1)}(r_1',r_2') = \mathcal{K}(r_1',r_2';\mathrm{i}k^{-1}\nabla_1',\mathrm{i}k^{-1}\nabla_2'). \tag{4.10}$$

Multiplying equation (4.9) by

$$Y_l^m(\theta_1,\phi_1)Y_{l'}^{m'*}(\theta_2,\phi_2)h_l^{(1)}(kr_1)h_{l'}^{(1)*}(kr_2) \tag{4.11}$$

with the spherical Hankel function of the first kind $h_l^{(1)}$, summing over all l, l', m, m', and using the expansion $(r'<r)$

$$\frac{\exp(\mathrm{i}k|r-r'|)}{|r-r'|} = 2\mathrm{i}k\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi')j_l(kr')h_l^{(1)}(kr) \tag{4.12}$$

we find that the field correlation tensor (3.1) vanishes everywhere outside the domain τ. We thus have shown that the far-zone condition (4.4) is sufficient for the source correlation to be non-radiating. We mention that the non-radiating current correlation tensor (3.6) fulfils this condition, as can be seen by inspection. In appendix 1 we present a generalisation of the condition (4.4) together with an alternative derivation which avoids the use of the expansions (4.8) and (4.12).

5. The non-existence of non-radiating surface field correlations

In the previous sections we have shown that non-trivial non-radiating three-dimensional stochastic current distributions do exist. Let us now discuss a related two-dimensional problem: is it possible to construct field correlations prescribed on some surface which do not radiate? The non-existence of such surface correlations is immediately deduced from the theory presented in § 4. The correlations of the tangential components of $\boldsymbol{E}$ and $\boldsymbol{B}$ on the boundary surface σ of some domain τ, together with the Sommerfeld radiation conditions for (3.1), uniquely determine the field correlations outside the domain. As we show below, the field correlations outside τ vanish if and only if the boundary correlations vanish almost everywhere on σ, i.e. non-trivial non-radiating boundary correlations do not exist.

In order to prove this statement, we recall that the EMF in free space can be derived from the solutions $\psi(\boldsymbol{r}, t)$ of the scalar wave equation (Wolf and Green 1953). As before we consider time harmonic fields, i.e. $\psi(\boldsymbol{r}, t) = \exp(-\mathrm{i}\omega t)\psi(\boldsymbol{r})$. To every prescribed boundary distribution $d(\boldsymbol{\sigma}) = \psi(\boldsymbol{r})|_{\boldsymbol{r}=\boldsymbol{\sigma}}$ with $\boldsymbol{\sigma}$ on the boundary surface there exists a unique equivalent stratum $\mu(\boldsymbol{\sigma})$ such that the unique solution of the exterior Dirichlet problem satisfying Sommerfeld's radiation condition at infinity reads

$$\int_\sigma \mu(\boldsymbol{\sigma})\frac{\partial}{\partial n}G(\boldsymbol{r}, \boldsymbol{\sigma}; k)\,\mathrm{d}\boldsymbol{\sigma} = \psi(\boldsymbol{r}). \tag{5.1}$$

For simplicity we have assumed here that k is not an eigenvalue of the corresponding interior problem. (If k is an eigenvalue, one has a slightly more complicated representation of the field (see e.g. Hönl *et al* (1961) and references therein), to which, however, the same analysis applies.) The above representation is easily transferred to the pertinent correlation $\langle\psi(\boldsymbol{r}_1)\psi^*(\boldsymbol{r}_2)\rangle$, $\langle d(\boldsymbol{\sigma}_1)d^*(\boldsymbol{\sigma}_2)\rangle$, and $\langle\mu(\boldsymbol{\sigma}_1)\mu^*(\boldsymbol{\sigma}_2)\rangle$.

Here we can start from a slightly more relaxed definition of non-radiating correlations $\langle d(\boldsymbol{\sigma}_1)d^*(\boldsymbol{\sigma}_2)\rangle$, namely that the radial part of the average Poynting vector decreases more rapidly than r^{-2}. The Poynting vector is known to be proportional to $\mathrm{Im}(\psi\boldsymbol{\nabla}\psi^*)$ (Wolf and Green 1953). Hence, from equations (4.1) and (5.1) we learn that non-radiating boundary correlations $\langle d(\boldsymbol{\sigma}_1)d^*(\boldsymbol{\sigma}_2)\rangle$ imply

$$\int_\sigma\int_\sigma \mathrm{d}\boldsymbol{\sigma}_1\,\mathrm{d}\boldsymbol{\sigma}_2\langle\mu(\boldsymbol{\sigma}_1)\mu(\boldsymbol{\sigma}_2)\rangle\exp[\mathrm{i}k(\boldsymbol{\sigma}_1.\boldsymbol{s}_1-\boldsymbol{\sigma}_2.\boldsymbol{s}_2)] = 0 \tag{5.2}$$

for any $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, where $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \sigma$.

Following the procedure of § 4, by virtue of equations (4.6) and (4.10), equation (5.2) can be cast into

$$\int_\sigma\int_\sigma \mathrm{d}\boldsymbol{\sigma}_1\,\mathrm{d}\boldsymbol{\sigma}_2\langle\mu(\boldsymbol{\sigma}_1)\mu(\boldsymbol{\sigma}_2)\rangle\frac{\partial}{\partial n_1}G(\boldsymbol{r}_1, \boldsymbol{\sigma}_1; k)\frac{\partial}{\partial n_2}G(\boldsymbol{r}_2, \boldsymbol{\sigma}_2; k) = 0 \tag{5.3}$$

for all $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ outside the domain τ. From equations (5.1) and (5.3) we conclude that $\langle\psi(\boldsymbol{r}_1)\psi^*(\boldsymbol{r}_2)\rangle$ as well as the related field correlations vanish almost everywhere outside τ. Hence, by continuity, the boundary field correlations, e.g. $\langle\boldsymbol{E}(\boldsymbol{\sigma}_1)\boldsymbol{E}^*(\boldsymbol{\sigma}_2)\rangle$, also vanish.

6. Discussion and generalisation

In this paper we have developed the theory of non-radiating sources of any degree of (first-order) coherence. The main results are the following.

(i) The construction of non-radiating current correlations as described by equations (3.4)–(3.6).

(ii) The sufficient and necessary condition (4.4) for non-radiating current correlations.

(iii) The non-existence of non-radiating field correlations prescribed on a surface as shown in § 5.

Let us briefly discuss the result (ii). The necessary and sufficient condition

$$\langle \tilde{\boldsymbol{j}}^{\mathrm{T}}(k\boldsymbol{s}_1)\tilde{\boldsymbol{j}}^{\mathrm{T}*}(k\boldsymbol{s}_2)\rangle = 0 \text{ for all directions } \boldsymbol{s}_1, \boldsymbol{s}_2$$

is the generalisation of the corresponding condition for deterministic current distributions, namely $\tilde{\boldsymbol{j}}^{\mathrm{T}}(k\boldsymbol{s}_1) = 0$ for all directions $\boldsymbol{s}_1$, established by Erber and Prastein (1970), Devaney and Wolf (1973) and rederived by Cohen and Bleistein (1977).

In this paper we have considered the possibility of non-radiating current correlations within the framework of first-order coherence theory. The generalisation to nth-order current correlations appears to be straightforward. We conjecture that the corresponding condition reads

$$\langle \tilde{\boldsymbol{j}}^{\mathrm{T}}(k\boldsymbol{s}_1)\ldots\tilde{\boldsymbol{j}}^{\mathrm{T}}(k\boldsymbol{s}_n)\tilde{\boldsymbol{j}}^{\mathrm{T}*}(k\boldsymbol{s}_{n+1})\ldots\tilde{\boldsymbol{j}}^{\mathrm{T}*}(k\boldsymbol{s}_{2n})\rangle = 0. \tag{6.1}$$

Equivalent to the condition (4.4) (with A1.4) is the set of conditions

$$\int_\tau\int_\tau \mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2\, j_l(kr_1)j_{l'}(kr_2)Y_l^{m*}(\theta_1,\phi_1)Y_{l'}^{m'}(\theta_2,\phi_2)\langle \boldsymbol{j}^{\mathrm{T}}(\boldsymbol{r}_1)\boldsymbol{j}^{\mathrm{T}*}(\boldsymbol{r}_2)\rangle = 0 \tag{6.2}$$

for all l, l', m, m'. These conditions lead to the interpretation that a stochastic current distribution is non-radiating if and only if the projections of the tensor $\langle \boldsymbol{j}^{\mathrm{T}}(\boldsymbol{r}_1)\boldsymbol{j}^{\mathrm{T}*}(\boldsymbol{r}_2)\rangle$ on the Hilbert space vectors $j_l(kr_1)j_{l'}(kr_2)Y_l^{m*}(\theta_1,\phi_1)Y_{l'}^{m'}(\theta_2,\phi_2)$ vanish for all l, l', m, m'. We thus find the following additional procedure for constructing non-radiating stochastic current distributions: consider an arbitrary tensor $\mathscr{X}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ and subtract its projections along the abovementioned Hilbert space vectors. The resulting tensor describes a non-radiating source. With respect to the related inverse source problem we learn that only the projection of $\langle \boldsymbol{j}^{\mathrm{T}}(\boldsymbol{r}_1)\boldsymbol{j}^{\mathrm{T}*}(\boldsymbol{r}_2)\rangle$ in the Hilbert space described above can be observed.

So far we have considered only stochastic current distributions which depend harmonically on time. However, non-radiating current correlations with a more general time dependence can also be constructed following a procedure similar to that outlined in § 3. Assume that the stochastic current $\boldsymbol{j}(\boldsymbol{r}, t)$ has a temporal Fourier decomposition whose spectrum contains a continuous part $\tilde{\boldsymbol{j}}(\boldsymbol{r}, k_{\mathrm{c}})$ as well as a discrete part $\tilde{\boldsymbol{j}}(\boldsymbol{r}, k_{\mathrm{d}})$, namely:

$$\boldsymbol{j}(\boldsymbol{r}, t) = \int \mathrm{d}k_{\mathrm{c}}\tilde{\boldsymbol{j}}(\boldsymbol{r}; k_{\mathrm{c}})\exp(-\mathrm{i}k_{\mathrm{c}}ct) + \sum_d \tilde{\boldsymbol{j}}(\boldsymbol{r}; k_{\mathrm{d}})\exp(-\mathrm{i}k_{\mathrm{d}}ct). \tag{6.3}$$

Equations (2.10), (2.11) and (3.3) show that the corresponding current correlation is non-radiating if

$$\int_\tau\int_\tau G(\boldsymbol{r}_1, \boldsymbol{r}_1'; k_1)G^*(\boldsymbol{r}_2, \boldsymbol{r}_2'; k_2)\langle \tilde{\boldsymbol{j}}^{\mathrm{T}}(\boldsymbol{r}_1'; k_1)\tilde{\boldsymbol{j}}^{\mathrm{T}*}(\boldsymbol{r}_2'; k_2)\rangle\,\mathrm{d}\boldsymbol{r}_1'\,\mathrm{d}\boldsymbol{r}_2' = 0 \tag{6.4}$$

with k_1, k_2 denoting any of the wavenumbers k_c or k_d. Consider now the class of current correlations defined as

$$\langle \tilde{j}^{\mathrm{T}}(\boldsymbol{r}_1; k_1)\tilde{\boldsymbol{j}}^{\mathrm{T}*}(\boldsymbol{r}_2; k_2)\rangle \equiv (\nabla_1^2 + k_1^2)(\nabla_2^2 + k_2^2)\mathscr{k}(\boldsymbol{r}_1, \boldsymbol{r}_2) \tag{6.5}$$

where $\mathscr{k}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ denotes a function which is everywhere twice continuously differentiable and which vanishes for $\boldsymbol{r}_1$ or $\boldsymbol{r}_2$ outside the source domain τ. Following the same procedure as outlined in § 3, we can show that the correlations (6.5) fulfil the condition (6.4) and thus describe a class of non-radiating sources.

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Note added in proof. For other recent results see Devaney A J 1979 *J. Math. Phys.* **20** 1687 (The inverse problem for random sources).

Appendix 1

The connection between the vanishing of the field correlation tensor in the far zone and everywhere outside the domain τ studied in § 4 can also be inferred from the following consideration. We notice that the relation

$$\nu(\boldsymbol{r}_1)\nu^*(\boldsymbol{r}_2) = \int_\sigma \mathrm{d}\boldsymbol{\sigma}_1\left[G(\boldsymbol{r}_1, \boldsymbol{\sigma}_1; k)\frac{\partial}{\partial n_1} - \frac{\partial}{\partial n_1}G(\boldsymbol{r}_1, \boldsymbol{\sigma}_1; k)\right]\nu(\boldsymbol{\sigma}_1)$$
$$\times \int_\sigma \mathrm{d}\boldsymbol{\sigma}_2\left[G(\boldsymbol{r}_2, \boldsymbol{\sigma}_2; k)\frac{\partial}{\partial n_2} - \frac{\partial}{\partial n_2}G(\boldsymbol{r}_2, \boldsymbol{\sigma}_2; k)\right]\nu(\boldsymbol{\sigma}_2) \tag{A1.1}$$

is valid for solutions $\nu(\boldsymbol{r}_j)$ of the scalar Helmholtz equations

$$(\nabla_j^2 + k^2)\nu(\boldsymbol{r}_j) = 0, \qquad j = 1, 2, \tag{A1.2}$$

with $\boldsymbol{r}_j$ in the domain τ bounded by the surface σ. Consider, for example, the electric-field tensor $\langle \boldsymbol{E}(\boldsymbol{r}_1)\boldsymbol{E}^*(\boldsymbol{r}_2)\rangle$ with $\boldsymbol{E}(\boldsymbol{r}_j)$ as given by equation (2.10). The vanishing of the far-field correlation tensor implies

$$\langle \boldsymbol{E}(\boldsymbol{r}_1)\boldsymbol{E}^*(\boldsymbol{r}_2)\rangle = \int_\tau\int_\tau \mathrm{d}\boldsymbol{r}_1'\,\mathrm{d}\boldsymbol{r}_2' G(\boldsymbol{r}_1, \boldsymbol{r}_1'; k)G^*(\boldsymbol{r}_2, \boldsymbol{r}_2'; k)\langle \boldsymbol{j}^{\mathrm{T}}(\boldsymbol{r}_1')\boldsymbol{j}^{\mathrm{T}*}(\boldsymbol{r}_2')\rangle = \mathrm{o}\left(\frac{1}{r_1 r_2}\right) \tag{A1.3}$$

with

$$\boldsymbol{j}^{\mathrm{T}}(\boldsymbol{r}_1) = (\mathbb{1} + k^{-2}\boldsymbol{\nabla}\boldsymbol{\nabla})\boldsymbol{j}(\boldsymbol{r}_1) \tag{A1.4}$$

and $r_j = |\boldsymbol{r}_j|$. On the other hand, the asymptotic expansion (4.1) shows that we also have the asymptotic relations

$$\frac{\partial}{\partial n_1}\langle \boldsymbol{E}(\boldsymbol{r}_1)\boldsymbol{E}^*(\boldsymbol{r}_2)\rangle = \mathrm{o}\left(\frac{1}{r_1 r_2}\right) \tag{A1.5}$$

$$\frac{\partial}{\partial n_2}\langle \boldsymbol{E}(\boldsymbol{r}_1)\boldsymbol{E}^*(\boldsymbol{r}_2)\rangle = \mathrm{o}\left(\frac{1}{r_1 r_2}\right) \tag{A1.6}$$

$$\frac{\partial^2}{\partial n_1 \partial n_2}\langle \boldsymbol{E}(\boldsymbol{r}_1)\boldsymbol{E}^*(\boldsymbol{r}_2)\rangle = \mathrm{o}\left(\frac{1}{r_1 r_2}\right). \tag{A1.7}$$

Multiplying (A1.1) by $\langle \boldsymbol{j}^{\mathrm{T}}(\boldsymbol{r}_1)\boldsymbol{j}^{\mathrm{T}*}(\boldsymbol{r}_2)\rangle$, integrating over $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, using equations (A1.3) and (A1.5)–(A1.7) and assuming that $\nu(\boldsymbol{\sigma}_j)$ is bounded uniformly as σ goes to infinity, we finally obtain

$$\int_\tau \int_\tau \nu(\boldsymbol{r}_1)\nu^*(\boldsymbol{r}_2)\langle \boldsymbol{j}^{\mathrm{T}}(\boldsymbol{r}_1)\boldsymbol{j}^{\mathrm{T}*}(\boldsymbol{r}_2)\rangle \,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2 = 0 \tag{A1.8}$$

as σ goes to infinity. Invoking expansion (4.12), choosing for the modes $\nu(\boldsymbol{r}_j)$ the set $j_l(kr_j)Y_l^m(\theta_j, \phi_j)$, and summing over l, l', m, m', the left-hand side of (A1.3) is shown to vanish everywhere outside the domain τ. Thus we have shown again that the vanishing of the field tensor in the far zone implies its vanishing everywhere outside the source domain.

Moreover, (A1.8) is a generalisation of the condition (4.4) with respect to the basis $\{\nu(\boldsymbol{r}_j)\}$ spanning the Hilbert space of the Helmholtz equations (A1.2). Our condition (4.4) is reproduced for the special choice $\nu(\boldsymbol{r}_j) = \exp(-\mathrm{i}k\boldsymbol{r}_j \,.\, \boldsymbol{s}_j)$. The generalisation (A1.8) of the crucial relation (4.4) is directly obtained from (4.4) by a transformation of the (assumedly) complete basis of the underlying Hilbert space.

Appendix 2

For the convenience of the reader, we rederive equation (2.7), following Jones (1964), from the vector analogue of Green's theorem for well-behaved vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ (Morse and Feshbach 1953):

$$\int_V \{(\nabla\times\nabla\times\boldsymbol{P}).\boldsymbol{Q} - \boldsymbol{P}.(\nabla\times\nabla\times\boldsymbol{Q})\}\,\mathrm{d}\tau = \int_\sigma \{(n\times\nabla\times\boldsymbol{P}).\boldsymbol{Q} + (\boldsymbol{n}\times\boldsymbol{P}).(\nabla\times\boldsymbol{Q})\}\,\mathrm{d}\tau \tag{A2.1}$$

and the definitions for the in-product, curl, and out-product of a tensor $\mathcal{Q}$ of the second rank and third order with a vector $\boldsymbol{P}$:

$$\boldsymbol{P}\times\mathcal{Q} = (\boldsymbol{P}\times\boldsymbol{Q}_x)\boldsymbol{i}_x + (\boldsymbol{P}\times\boldsymbol{Q}_y)\boldsymbol{i}_y + (\boldsymbol{P}\times\boldsymbol{Q}_z)\boldsymbol{i}_z \tag{A2.2}$$

$$\nabla\times\mathcal{Q} = (\nabla\times\boldsymbol{Q}_x)\boldsymbol{i}_x + (\nabla\times\boldsymbol{Q}_y)\boldsymbol{i}_y + (\nabla\times\boldsymbol{Q}_z)\boldsymbol{i}_z \tag{A2.3}$$

$$\boldsymbol{P}.\mathcal{Q} = (\boldsymbol{P}.\boldsymbol{i}_x)\tilde{\boldsymbol{Q}}_x + (\boldsymbol{P}.\boldsymbol{i}_y)\tilde{\boldsymbol{Q}}_y + (\boldsymbol{P}.\boldsymbol{i}_z)\tilde{\boldsymbol{Q}}_z. \tag{A2.4}$$

The vectors $\boldsymbol{i}_x$, $\boldsymbol{i}_y$ and $\boldsymbol{i}_z$ are the unit vectors of a Cartesian coordinate system whereas the tensor $\mathcal{Q}$ is written in dyadic notation in terms of the vectors $\boldsymbol{Q}_x$, $\boldsymbol{Q}_y$, $\boldsymbol{Q}_z$, or $\tilde{\boldsymbol{Q}}_x$, $\tilde{\boldsymbol{Q}}_y$, $\tilde{\boldsymbol{Q}}_z$:

$$\mathcal{Q} = \boldsymbol{Q}_x\boldsymbol{i}_x + \boldsymbol{Q}_y\boldsymbol{i}_y + \boldsymbol{Q}_z\boldsymbol{i}_z = \boldsymbol{i}_x\tilde{\boldsymbol{Q}}_x + \boldsymbol{i}_y\tilde{\boldsymbol{Q}}_y + \boldsymbol{i}_z\tilde{\boldsymbol{Q}}_z. \tag{A2.5}$$

Taking for $\boldsymbol{Q}$ in equation (A2.1) $\boldsymbol{Q}_x$, $\boldsymbol{Q}_y$ and $\boldsymbol{Q}_z$ respectively, multiplication from the right with $\boldsymbol{i}_x$, $\boldsymbol{i}_y$ and $\boldsymbol{i}_z$ and addition of the resulting equation together with (A2.2) and (A2.3) leads to equation (2.7).

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